Some Characterizations of TTC in Multi-Object Reallocation Problems

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Shift Exchange

- A firm assigns shifts to its employees:

	Mon	Tue	Wed	Thu	Fri
am	Alice	Carol	Bob	Carol	Bob
pm	Bob	Alice	Alice	Alice	Carol

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- Reallocating the shifts could make all workers happier.

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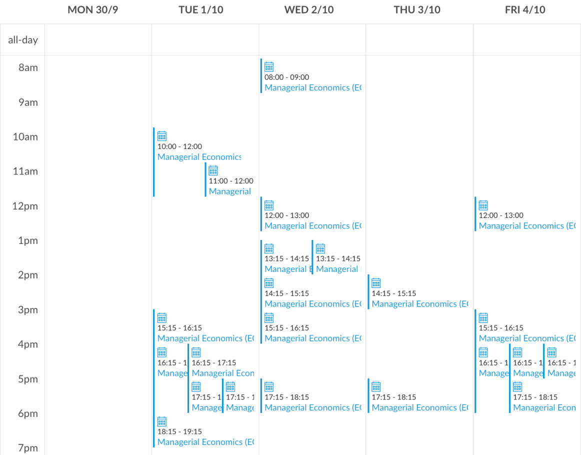
- Reallocating the shifts could make all workers happier.
- How, then, should trades be organized?

Shift Exchange

Managerial Economics Tutorial Schedule



Reallocation problems

Shift Exchange is an instance of multi-object reallocation without transfers:

- a group of agents, each of whom
  - initially owns a set of *heterogeneous* and *indivisible* objects.
  - has strict preferences over *bundles* of objects.
  - cares only about her own assigned bundle.
- no restrictions on trade, i.e., all allocations are admissible.
- a generalization of the "housing market" (Shapley and Scarf, 1974).

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Other instances include:

- course (re)allocation (Budish, 2011), tuition and student exchange (Dur and Ünver, 2019; Andersson et al., 2021), living-donor kidney exchange (Roth et al., 2005, 2004).

Two challenges

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(2) Conflict among "ideal" properties

- Pareto efficiency, individual rationality, and strategy-proofness are incompatible (Sönmez, 1999).
- we circumvent the incompatibility by relaxing Pareto efficiency and strategy-proofness.

Our contribution

- Our main result is a characterization of TTC under "responsive" preferences: it is the only individual-good-based rule satisfying balancedness together with
  - individual-good efficiency
  - individual rationality, and
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- The upshot: TTC performs surprisingly well according to the three criteria of interest.

Related Literature

Related models of multi-unit reallocation

- Altuntaş et al. (2023): lexicographic preferences
- Biró et al. (2022): multi-unit housing market
- Manjunath and Westkamp (2021): trichotomous preferences
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Single-unit reallocation

- Shapley and Scarf (1974), Ma (1994)
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We owe the largest debt to Altunta\u0219 et al. (2023), who proved

- TTC is drop strategy-proof
- the first characterization of TTC

Outline

1. Setup
2. Lexicographic preferences
3. Responsive preferences
4. The Housing Market
5. Related Literature
6. Conditionally lexicographic preferences
7. Conclusion

Model: Preliminaries

A problem consists of:

- a set $N = \{1, 2, \dots, n\}$ of agents
- a set O of *heterogeneous* and *indivisible* objects, with $|O| \geq n$.
- an initial allocation $\omega = (\omega_i)_{i \in N}$ of objects to agents s.th.
  - $\omega_i \cap \omega_j = \emptyset$ when $i \neq j$
  - $\bigcup_{i \in N} \omega_i = O$
  - ω_i is agent i's (nonempty) endowment
- a profile $P = (P_i)_{i \in N}$ of strict preferences over bundles, 2^O
  - each P_i belongs to some domain $\mathcal{P}$
  - R_i is the associated "*at least as good as*" relation

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N, O, ω are fixed, so we identify a problem with its profile P.

Thus, $\mathcal{P}^N$ is the set of all problems.

Model: Allocations and rules

- An allocation $\mu = (\mu_i)_{i \in N}$ is a (re)assignment of objects to agents s.th.
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- $\mathcal{A}$ denotes the set of allocations
- A rule (on $\mathcal{P}$) is a systematic procedure for reallocating the objects, i.e., a function $\varphi : \mathcal{P}^N \to \mathcal{A}$.

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Lexicographic Preferences

- Agent i's preferences P_i are lexicographic if for any distinct bundles X and Y,
  - if i prefers the best object in X to that in Y, then $X \; P_i \; Y$;
  - if these objects are the same, then i compares the second-best object in X to that in Y, and so on.
  - if $X \supsetneq Y$, then $X \; P_i \; Y$.
- Let $\mathcal{L}$ denote the lexicographic domain.
- Any $P_i \in \mathcal{L}$ is identified by its ranking over singletons e.g., $P_i : o_1, o_2, \ldots, o_m$ means $P_i \in \mathcal{L}$ and $o_1 \; P_i \; o_2 \; P_i \cdots P_i \; o_m$.

Top Trading Cycles

For each profile P, the TTC rule selects the allocation $\varphi^{\mathsf{TTC}}(P)$ obtained as follows.

TTC (P)

For each step $t \geq 1$,

- Each agent points to her top-ranked remaining object.
- Each object points to its owner.
- All cycles are "executed."
- Remove all objects (but not the agents) involved in a cycle.
- If no objects remain, stop and return the allocation.

Top Trading Cycles

Consider following problem (with endowments in red):

P_1	P_2	P_3
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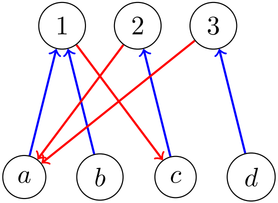

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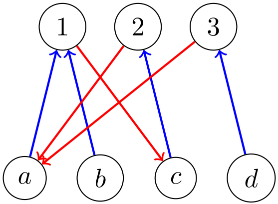
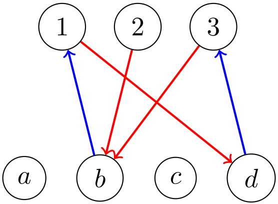


Step 2.



$$\varphi^{\mathsf{TTC}}(P) = (\{c, d\}, \{a\}, \{b\})$$

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4. the worst endowment lower bound if, for each profile P and each agent i,
$$\text{for all } o \in \varphi_i(P), \quad o\ R_i \min_{P_i}(\omega_i).$$
   - e.g., if $P_i : a, b, x, c, d, y, e$ and $\omega_i = \{x, y\}$, then $\varphi_i(P)$ does not contain e.

Properties: II (Incentives)

Given agent i's true preference P_i, we say that

- P_i' is a drop strategy if it is obtained by dropping an object in $O \backslash \omega_i$ to the bottom.
- P_i^* is a truncation strategy if it is obtained by dropping a "tail subset" of $O \backslash \omega_i$ to the bottom.[1]

[1] i.e., a subset X such that if $x \in X$, $y \in O \backslash \omega_i$, and $x \, P_i \, y$, then $y \in X$.

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Example

Suppose $P_i : a, b, x, c, d, y, e$ and $\omega_i = \{x, y\}$. Then:

- $P_i' : b, x, c, d, y, e, a$ is obtained by dropping object a.
- $P_i^* : a, b, x, c, y, d, e$ is obtained by "truncating at c" i.e., dropping the set $\{o \in O\backslash\omega_i \mid c\, P_i\, o\} = \{d, e\}$.
- $P_i^\circ : a, x, y, b, c, d, e$ is obtained by "truncating at a" i.e., dropping the set $\{o \in O\backslash\omega_i \mid a\, P_i\, o\} = \{b, c, d, e\}$.

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A rule φ is

1. drop strategy-proof if no agent can manipulate via *drop strategies*.
2. truncation-proof if no agent can manipulate via *truncation strategies*.
3. strategy-proof if no agent can manipulate via *any strategies*.

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Properties of TTC

Proposition

On the lexicographic domain, TTC satisfies

1. *Pareto efficiency*,
2. *balancedness*,
3. *individual rationality*,
4. *the worst endowment lower bound*,
5. *truncation-proofness*,
6. *drop strategy-proofness*.

Two characterizations

Theorem

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Independence of properties

- Pareto efficiency: no-trade rule
- worst endowment lower bound: serial dictatorships subject to balancedness
- balancedness: serial dictatorships subject to worst endowment lower bound
- truncation-proofness / drop strategy-proofness: straightforward.

Independence of properties

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Remark

On the lexicographic domain, there are other rules satisfying

- balancedness,
- Pareto efficiency,
- individual rationality, and
- truncation-proofness.

Discussion: Properties

Balancedness: for each profile P and each agent i, $|\varphi_i(P)| = |\omega_i|$.

- an inviolable constraint in many practical problems:
  - in shift reallocation, it may be imposed for training reasons
  - a requirement in student exchange programs (e.g., Erasmus, The Tuition Exchange)
- in the absence of constraints, it has some normative appeal:
  - simplicity: balanced allocations can be obtained from single-object exchanges.
  - a mild form of equity

Discussion: Properties

The worst endowment lower bound: for each profile P and each agent i, $\varphi_i(P) \subseteq \{o \in O \mid o \, R_i \min_{P_i}(\omega_i)\}$.

- agrees with individual rationality for single-object problems:
  - one possible extension to multi-object problems.
- restricts the set of objects that can make up an agent's bundle
  - under individual rationality, an agent can be assigned *any object* if part of a desirable bundle.

Discussion: Properties

Truncation-proofness: no agent can manipulate via *truncation strategies*.

- coupled with worst endowment lower bound, it ensures agents cannot benefit by "vetoing" objects they do not own.
- truncations are compelling and simple to implement
  - agents need only identify cutoff object
  - very close to true preferences (they agree on $O \backslash \omega_i$ and on ω_i).
  - in many settings, truncations are "exhaustive" (Roth and Rothblum, 1999; Ehlers, 2008; Kojima and Pathak, 2009; Kojima, 2013).
  - hence, a minimal requirement.

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Responsive preferences

- Agent i's preferences P_i are responsive if
  for any bundle X and any $y, z \in O \backslash X$,

$$(X \cup y)\, P_i\, (X \cup z) \iff y\, P_i\, z.$$

- Let $\mathcal{R}$ denote the responsive domain. Note that $\mathcal{L} \subseteq \mathcal{R}$.

- Given $P_i \in \mathcal{R}$, let $\succ^{P_i}$ denote the associated rank-order list over O.

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Remark.

There are many "responsive extensions" of a rank-order list $\succ^{P_i}$.

For example, it is possible that

$$\succ^{P_i} = \succ^{P_i'} : a, b, c, d$$

even though

$$\{a, d\}\, P_i\, \{b, c\} \text{ and } \{b, c\}\, P_i'\, \{a, d\}.$$

Simple rules

- We focus on rules that depend only on the orderings $\succ^{P} = (\succ^{P_i})_{i \in N}$ associated with a profile $P = (P_i)_{i \in N}$.
- Formally, a rule φ is individual-good-based if

$$\text{for all } P, P' \in \mathcal{R}^N, \;\; \succ^{P} = \succ^{P'} \implies \varphi(P) = \varphi(P').$$

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- This assumption is common—in theory and in practice.
  - e.g., in the National Resident Matching Program, hospitals report only their rank-order lists over individual doctors (Milgrom, 2009, 2011).
- TTC is an individual-good-based rule.

Properties: III

Our properties are defined as before, with the understanding that *drop strategies* and *truncation strategies* for P_i are defined wrt $\succ^{P_i}$.

Example

Suppose P_i is such that $\succ^{P_i}$: a, b, x, c, d, y, e and $\omega_i = \{x, y\}$. Then:

- (any P_i' with) $\succ^{P_i'}$: b, x, c, d, y, e, a is obtained by dropping object a.
- (any P_i^* with) $\succ^{P_i^*}$: a, b, x, c, y, d, e is obtained by "truncating at c".

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Proposition

TTC is not drop strategy-proof, but it is truncation-proof.

Properties: Efficiency

- The restriction to individual-good-based is substantive
- Consider the following problem (with endowments in red):

$\succ^{P_1}$	$\succ^{P_2}$
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- Complete swap $(\{b,c\},\{a,d\})$ is the unique Pareto efficient + individually rational allocation iff

$$\{b,c\}\, P_1\, \{a,d\} \quad \text{and} \quad \{a,d\}\, P_2\, \{b,c\}\,.$$

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$\Longrightarrow$ No individual-good-based rule is Pareto efficient + individually rational (Manjunath and Westkamp, 2024)

Properties: Efficiency

A rule φ is individual-good efficient (ig-efficient) if, for each profile P, $\varphi(P)$ does not admit a Pareto-improving single-object exchange at P.[2]

[2] i,.e., a cycle

$$C = (i_0, o_1, i_1, \ldots, i_{k-1}, o_k, i_k = i_0)$$

such that, for all $\ell \in \{0, \ldots, k-1\}$,

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Proof.

- Let φ be an individual-good-based rule satisfying properties (1)-(4).
- By our theorem for lexicographic prefs., φ agrees with φ^{TTC} on $\mathcal{L}^N$.
- Let $P \in \mathcal{R}^N$, and let $P' \in \mathcal{L}^N$ be such that $\succ^{P'} = \succ^{P}$.
- Because φ and φ^{TTC} are individual-good-based,

$$\varphi(P) = \varphi(P') = \varphi^{\mathsf{TTC}}(P') = \varphi^{\mathsf{TTC}}(P). \quad \square$$

Two characterizations

Theorem

An individual-good-based rule satisfies

1. *balancedness,*
2. *ig-efficiency,*
3. *the worst endowment lower bound, and*
4. *truncation-proofness*

if and only if it is TTC.

Lemma

If an individual-good-based rule is balanced and individually rational, then it satisfies the worst endowment lower bound.

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Lemma

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Results: Incentives

Though it is manipulable, we can show that TTC is

- maxmin strategy-proof; and
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That is, for any problem P, any agent i, any u_i that represents P_i, and any report P_i',

$$\min_{P_{-i}} u_i \left(\varphi_i^{\mathsf{TTC}} (P_i, P_{-i}) \right) \geq \min_{P'_{-i}} u_i \left(\varphi_i^{\mathsf{TTC}} (P'_i, P'_{-i}) \right)$$

$$\max_{P_{-i}} u_i \left(\varphi_i^{\mathsf{TTC}} (P_i, P_{-i}) \right) \geq \max_{P'_{-i}} u_i \left(\varphi_i^{\mathsf{TTC}} (P'_i, P'_{-i}) \right).$$

Outline

1. Setup
2. Lexicographic preferences
3. Responsive preferences
4. The Housing Market
5. Related Literature
6. Conditionally lexicographic preferences
7. Conclusion

The Housing Market

- The housing market is the special case in which each agent owns and receives one object.
- In this model:
  - only TTC is Pareto efficient, individually rational, and strategy-proof (Ma, 1994).
  - all allocations are balanced.
  - the worst endowment lower bound coincides with individual rationality.

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Corollary

Only TTC is Pareto efficient, individually rational, and truncation-proof.

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- In this model:
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  - all allocations are balanced.
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Corollary

Only TTC is Pareto efficient, individually rational, and truncation-proof.

- Though a planner with a stake in the outcome may consider relaxing strategy-proofness to truncation-proofness ...
- ... this relaxation does not give rise to any new rules.

Proof sketch

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Alternative models and allocation rules

Domain	Rule	ig-based	ig-EFF	truncation proof	IR	Pareto efficient	strategy proof
responsive	SD	✓	✓	✓	✗	✓	✓
	STC	✓	✗	✓	✓	✗	✓
	TTC	✓	✓	✓	✓	✗	✗
dichotomous	Priority	✓	✓	✓	✓	✓	✓
trichotomous	CIRP	✓	✓	✓	✓	✓	✓
combinatorial	approx. CE	✗	✓	–	✓	✓	✗

Notes:

- SD = Serial/sequential dictatorships (e.g., Ehlers and Klaus (2003); Hatfield (2009))
- STC = Segmented Trading Cycles (Pápai, 2003)
- Priority = Priority Mechanisms (Andersson et al., 2021)
- CIRP = Component-wise IR Priority rules (Manjunath and Westkamp, 2021)
- approx. CE = approximate competitive equilibrium (e.g., Echenique et al., 2023)

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Conditionally lexicographic preferences

- Agent i's preferences P_i are conditionally lexicographic if for any bundle $Y \subsetneq O$ and any nonempty $X \subseteq O \backslash Y$, there is an object $\max_{P_i}(X \mid Y) \in X$ which is "lexicographically best among X conditional on receiving Y."
  - $\mathcal{CL} \cap \mathcal{R} = \mathcal{L}$, where $\mathcal{CL}$ denotes the conditionally lexicographic domain.

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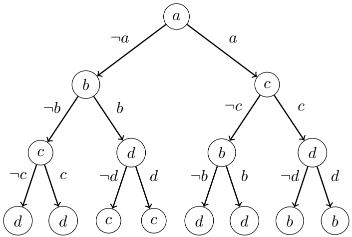
  - $\mathcal{CL} \cap \mathcal{R} = \mathcal{L}$, where $\mathcal{CL}$ denotes the conditionally lexicographic domain.

- Conditionally lexicographic preferences
  - permit *complementarity* between objects.
  - simple reporting language in terms of "preference trees."



Properties

Our properties are the same, except for two modifications:

- the worst endowment lower bound posits that, for each profile P and each agent i, $\varphi_i(P)$ does not contain objects that are "conditionally worse" than all objects in her endowment (conditional on receiving $\varphi_i(P)$).
- drop strategy-proofness posits that no agent can manipulate by "*dropping an object to the bottom of her lexicographic preference tree.*"

A characterization

- The extension of TTC to the conditionally lexicographic domain is called Augmented Top Trading Cycles (ATTC) (Fujita et al., 2018)
  - at step t, agent i points to $\max_{P_i} \left(O^t \mid \mu_i^{t-1}\right)$, where O^t is the set of remaining objects and μ_i^{t-1} is i's assignment after step $t-1$.
  - *not* individual-good-based as it uses information contained in preference trees.

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  - *not* individual-good-based as it uses information contained in preference trees.

Theorem

On the conditionally lexicographic domain, only ATTC satisfies

- *Pareto efficiency*
- *balancedness*
- *the worst endowment lower bound, and*
- *drop strategy-proofness.*

Maximal domain results

- It is known that ig-efficiency = Pareto efficiency on the lexicographic domain (Aziz et al., 2019).
- The conditionally lexicographic domain is similarly appealing.

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Proposition

1. *ig-efficiency = Pareto efficiency on the conditionally lexicographic domain.*
2. *$\mathcal{CL}$ is a maximal domain on which ig-efficiency = Pareto efficiency.*

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Conclusion

- Our axiomatic analysis helps us to better understand the trade-offs involved in multi-object reallocation.
- Although it is manipulable, TTC performs surprisingly well according to three criteria of interest: efficiency, individual rationality, and incentives.

Thank you!

☺

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Proof Sketch. Step 1: Select a "minimal profile"

- Toward contradiction, suppose $\varphi \neq \varphi^{\mathsf{TTC}}$ is Pareto efficient, individually rational, and truncation-proof.
- We select a profile P which is "minimal" according to some criteria—for that we need some notation.

[3] If multiple cycles obtain, we select one with a fixed tie-break rule.

Proof Sketch. Step 1: Select a "minimal profile"

- Toward contradiction, suppose $\varphi \neq \varphi^{\mathsf{TTC}}$ is Pareto efficient, individually rational, and truncation-proof.
- We select a profile P which is "minimal" according to some criteria—for that we need some notation.
- For each "conflict profile" $P \in \mathcal{C} := \left\{P' \mid \varphi(P') \neq \varphi^{\mathsf{TTC}}(P')\right\}$, let
  - $C_t(P)$ be the cycle executed at step t of $\mathsf{TTC}(P)$.[3]
  - $s(P) = \sum_{i \in N} |\{o \in O \mid o\, R_i\, o_i\}|$ be the size of P, where $\omega_i = \{o_i\}$.
  - $\rho(P) = \min\{t \in \mathbb{N} \mid \varphi(P) \text{ does not execute } C_t(P)\}$.
- Let $t := \min_{P \in \mathcal{C}} \rho(P)$ be the "earliest point of departure between φ and φ^{TTC} across all conflict profiles."
- Among all profiles in $\{P' \in \mathcal{C} \mid \rho(P') = t\}$, let P be one that minimizes $s(P)$.

[3] If multiple cycles obtain, we select one with a fixed tie-break rule.

Step 2: Agents on $C_t(P)$ retain their endowments

- Because $\rho(P) = t$, $\varphi(P)$ executes cycles $C_1(P), \ldots, C_{t-1}(P)$ but not $C_t(P)$.
- Let $C \coloneqq C_t(P)$, say

$$C = (i_0, o_1, i_1, o_2, \ldots, i_{k-1}, o_k, i_k = i_0).$$

- Because $\varphi(P)$ does not execute C, can assume WLOG that i_k $(= i_0)$ does not receive o_1. Thus, $\varphi^{\mathsf{TTC}}_{i_k}(P) = o_1 \; P_{i_k} \; \varphi_{i_k}(P)$.[4]

[4]By individual rationality, the number of agents on C is $k \geq 2$.

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- Thus, the profile P looks as follows (endowments are blue):

P_{i_1}	P_{i_2}	$\cdots$	$P_{i_{k-1}}$	P_{i_k}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_2	o_3	$\cdots$	o_k	o_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_1	o_2	$\cdots$	o_{k-1}	$\varphi_{i_k}(P)$

[4] By individual rationality, the number of agents on C is $k \geq 2$.

Step 2: Agents on $C_t(P)$ retain their endowments

- Suppose $\varphi_{i_k}(P) \neq o_k$.
- By individual rationality, the profile P looks as follows:

$$
\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_1 & o_2 & \cdots & o_{k-1} & \varphi_{i_k}(P) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & o_k \\
 & & & & \vdots
\end{array}
$$

Step 2: Agents on $C_t(P)$ retain their endowments

- Suppose $\varphi_{i_k}(P) \neq o_k$.
- Let P'_{i_k} be the truncation of P_{i_k} at o_1:

$$\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P'_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & o_k \\
o_1 & o_2 & \cdots & o_{k-1} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \varphi_{i_k}(P) \\
 & & & & \vdots
\end{array}$$

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P_{i_1}	P_{i_2}	$\cdots$	$P_{i_{k-1}}$	P'_{i_k}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_2	o_3	$\cdots$	o_k	o_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	o_k
				$\vdots$
o_1	o_2	$\cdots$	o_{k-1}	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\varphi_{i_k}(P)$
				$\vdots$

- Letting $P' := \left(P'_{i_k}, P_{-i_k}\right)$, our choice of P implies that $\varphi(P')$ executes cycles $C_1(P'), \ldots, C_t(P')$ $(= C_1(P), \ldots, C_t(P))$.
- Thus, $\varphi_{i_k}(P') = o_1 \; P_{i_k} \; \varphi_{i_k}(P)$, a violation of truncation-proofness.

Step 2: Agents on $C_t(P)$ retain their endowments

- Thus, $\varphi_{i_k}(P) = o_k$, which means that $o_k \, P_{i_{k-1}} \, \varphi_{i_{k-1}}(P)$.

$$
\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_1 & o_2 & \cdots & \varphi_{i_{k-1}}(P) & \varphi_{i_k}(P) = o_k \\
\vdots & \vdots & \ddots & \vdots & \vdots
\end{array}
$$

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- Thus, $\varphi_{i_k}(P) = o_k$, which means that $o_k \, P_{i_{k-1}} \, \varphi_{i_{k-1}}(P)$.
- If $\varphi_{i_{k-1}}(P) \neq o_{k-1}$, then the profile P looks as follows:

$$
\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
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\vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & o_{k-1} & \\
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\end{array}
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\vdots & \vdots & \ddots & \vdots & \vdots \\
o_1 & o_2 & \cdots & \varphi_{i_{k-1}}(P) & \varphi_{i_k}(P) = o_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & o_{k-1} & \\
 & & & \vdots &
\end{array}$$

- A similar argument shows that $\varphi_{i_{k-1}}(P) = o_{k-1}$.

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- Thus, $\varphi_{i_k}(P) = o_k$, which means that $o_k \, P_{i_{k-1}} \, \varphi_{i_{k-1}}(P)$.
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$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_2	o_3	$\cdots$	o_k	o_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_1	o_2	$\cdots$	$\varphi_{i_{k-1}}(P) = o_{k-1}$	$\varphi_{i_k}(P) = o_k$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$

- A similar argument shows that $\varphi_{i_{k-1}}(P) = o_{k-1}$.

Step 2: Agents on $C_t(P)$ retain their endowments

- By a recursive argument, the profile P looks as follows:

$$
\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
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$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_2	o_3	$\cdots$	o_k	o_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_1	$\varphi_{i_2}(P) = o_2$	$\cdots$	$\varphi_{i_{k-1}}(P) = o_{k-1}$	$\varphi_{i_k}(P) = o_k$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$

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\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\varphi_{i_1}(P) = o_1 & \varphi_{i_2}(P) = o_2 & \cdots & \varphi_{i_{k-1}}(P) = o_{k-1} & \varphi_{i_k}(P) = o_k \\
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\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\varphi_{i_1}(P) = o_1 & \varphi_{i_2}(P) = o_2 & \cdots & \varphi_{i_{k-1}}(P) = o_{k-1} & \varphi_{i_k}(P) = o_k \\
\vdots & \vdots & \ddots & \vdots & \vdots
\end{array}
$$

- ... but then φ is not Pareto efficient! Back □