Characterizing TTC via endowments-swapping-proofness and truncation-proofness

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Abstract

In the object reallocation problem introduced by Shapley and Scarf (1974), Fujinaka and Wakayama (2018) showed that Top Trading Cycles (TTC) is the unique rule satisfying *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness*. We show that the uniqueness remains true if *strategy-proofness* is weakened to *truncation-proofness*.

Keywords: housing markets; Top Trading Cycles; endowment manipulation; truncation-proofness.

JEL Classification: C78; D47; D71.

1 Introduction

We consider the *object reallocation problem* introduced by Shapley and Scarf (1974). There is a group of agents, each of whom is endowed with a distinct object and equipped with strict preferences over all objects. An allocation is any redistribution of objects such that each agent receives one object. A *rule* specifies how objects are redistributed given the agents' endowments and their reported preferences.

Ma (1994) showed that only Gale's *Top Trading Cycles (TTC)* rule satisfies *individual rationality*, *strategy-proofness*, and *Pareto efficiency*. Recent papers have shown that the uniqueness remains true under substantially weaker criteria. For example, Ekici (2024) demonstrated that *Pareto efficiency* can be weakened to *pair efficiency*, and Coreno and Feng (2024) established

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that *strategy-proofness* can be relaxed to *truncation-proofness*.[1] In another direction, Fujinaka and Wakayama (2018) provided an alternative characterization by replacing *Pareto efficiency* with a (logically unrelated) incentive property, *endowments-swapping-proofness*.

In this note we characterize TTC through *individual rationality*, *truncation-proofness*, and *endowments-swapping-proofness*. Thus, we generalize the result of Fujinaka and Wakayama (2018) by weakening *strategy-proofness* to *truncation-proofness*. Additionally, we show that the result of Ekici (2024) cannot be generalized in the same way: there are other rules satisfying *individual rationality*, *truncation-proofness*, and *pair efficiency*.

2 Preliminaries

Let $N \coloneqq \{1, \ldots, n\}$ be a finite set of *agents*, and O a set of *objects* with $|O| = n$. An *allocation* is a bijection $\mu : N \to O$. Let $\mathcal{A}$ denote the set of allocations. For each $\mu \in \mathcal{A}$ and each $i \in N$, μ_i denotes agent i's *assignment* at μ, i.e., $\mu_i = \mu(i)$. Let $P = (P_i)_{i \in N}$ be a preference profile over O, where P_i denotes the (strict) preference of agent i. The weak preference relation associated with P_i is denoted by R_i.[2] Let $\mathcal{P}$ be the set of all strict preferences. We use the standard notation (P'_i, P_{-i}) to denote the profile obtained from P by replacing agent i's preference relation P_i with $P'_i \in \mathcal{P}$. A *problem* is a pair $(\omega, P) \in \mathcal{A} \times \mathcal{P}^N$, where $\omega = (\omega_i)_{i \in N}$ is an *initial allocation*. For each $i \in N$, we say that object ω_i is agent i's *endowment* or that agent i is the *owner* of object ω_i. A *rule* is a function $f : \mathcal{A} \times \mathcal{P}^N \to \mathcal{A}$ that associates with each problem (ω, P) an allocation $f(\omega, P)$. For each $i \in N$, $f_i(\omega, P)$ denotes agent i's assignment at $f(\omega, P)$. Let (ω, P) be a problem and $i, j \in N$. Denote by ω^{ij} the initial allocation obtained from ω by letting agents i and j swap their endowments.[3] We say that $P'_i \in \mathcal{P}$ is a *truncation strategy* for (ω_i, P_i) if (i) $\{o \in O \mid o \, P'_i \, \omega_i\} \subseteq \{o \in O \mid o \, P_i \, \omega_i\}$, and (ii) P'_i agrees with P_i on $O \setminus \{\omega_i\}$, i.e., $P'_i|_{O \setminus \{\omega_i\}} = P_i|_{O \setminus \{\omega_i\}}$.[4] Moreover, P'_i is the *truncation of* (ω_i, P_i) *at* x if, in addition, $\{o \in O \mid o \, P'_i \, \omega_i\} = \{o \in O \mid o \, R_i \, x\}$ (i.e., P'_i ranks ω_i immediately below object x). Denote the set of all truncation strategies for (ω_i, P_i) by $\mathcal{T}(\omega_i, P_i)$.

We introduce four properties of rules that are central to our analysis. A rule f is

individually rational if, for each (ω, P) and each i, $f_i(\omega, P) \, R_i \, \omega_i$.

truncation-proof if, for each (ω, P), each i, and each $P'_i \in \mathcal{T}(\omega_i, P_i)$, $f_i(\omega, P) R_i f_i(\omega, (P'_i, P_{-i}))$.

endowments-swapping-proof if, for each (ω, P), there is no pair $\{i, j\}$ of agents such that $f_i(\omega^{ij}, P) \, P_i \, f_i(\omega, P)$ and $f_j(\omega^{ij}, P) \, P_j \, f_j(\omega, P)$.

[1] A rule is *truncation-proof* if no agent can manipulate by "truncating" her list of acceptable objects, i.e., elevating her own object in her preference list while preserving the original ordering of all other objects.

[2] That is, for all $a, b \in O$, $a \, R_i \, b$ means that $a \, P_i \, b$ or $a = b$.

[3] That is, $\omega^{ij} \in \mathcal{A}$ is such that $\omega^{ij}_i = \omega_j$, $\omega^{ij}_j = \omega_i$, and, for each $k \in N \setminus \{i, j\}$, $\omega^{ij}_k = \omega_k$.

[4] For each $X \subseteq O$, $P_i|_X$ is the restriction of P_i to X. Formally, $P_i|_X = P_i \cap (X \times X)$.

pair-efficient if, for each (ω, P), there is no pair $\{i, j\}$ of agents such that $f_i(\omega, P)\ P_j\ f_j(\omega, P)$ and $f_j(\omega, P)\ P_i\ f_i(\omega, P)$.

Top Trading Cycles

Let φ denote the *Top Trading Cycles (TTC) rule*. For each problem (ω, P), $\varphi(\omega, P)$ is the allocation determined by the following *TTC algorithm* at (ω, P), which we call $\text{TTC}(\omega, P)$.

Algorithm: $\text{TTC}(\omega, P)$.

Step τ (≥ 1)**:** Each agent points to her most-preferred remaining object given P. Each remaining object points to its owner given ω. There exists at least one *cycle*. *Execute* all cycles by assigning each agent involved in a cycle the object to which she points. Remove all objects involved in a cycle. If some objects remain, then proceed to step $\tau + 1$.

Termination: The algorithm terminates (in at most n steps) when no object remains.

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3 The main result

Theorem 1. *A rule f is* ***individually rational****,* ***truncation-proof****, and* ***endowments-swapping-proof*** *if and only if $f = \varphi$.*

Proof of Theorem 1

It suffices to prove the uniqueness (only if) part of the theorem. Toward contradiction, suppose that f satisfies the stated properties but $f \neq \varphi$. We start by selecting a problem which is "minimal" according to some criteria. As in Coreno and Feng (2024), we simultaneously exploit the notions of "size" from Sethuraman (2016) and "similarity" from Ekici (2024).

Size: The *size* of a problem (ω, P) is $s(\omega, P) = \sum_{i \in N} |\{o \in O \mid o\, R_i\, \omega_i\}|$.

For each problem (ω, P) and each $t \in \mathbb{N}$, let $\mathcal{C}_t(\omega, P)$ be the set of cycles that obtain at step t of $\text{TTC}(\omega, P)$.[5] For any cycle C, let $N(C)$ and $O(C)$ be the sets of agents and objects, respectively, that are involved in C. We say that an allocation μ *executes* C if, for each $i \in N(C)$, μ_i is the object to which i points on C; otherwise, we say that μ *does not execute* C.

Similarity: The *similarity* between f and φ is a function $\rho : \mathcal{A} \times \mathcal{P}^N \to \{1, \ldots, n+1\}$ defined as follows. For each problem (ω, P), if $f(\omega, P) = \varphi(\omega, P)$, then $\rho(\omega, P) = n + 1$; otherwise,

$$\rho(\omega, P) = \min \left\{ \tau \in \{1, \ldots, n\} \mid \text{there exists } C \in \mathcal{C}_\tau(\omega, P) \text{ such that } f(\omega, P) \text{ does not execute } C \right\}.$$

That is, $\rho(\omega, P) = \tau$, where τ is the earliest step of $\text{TTC}(\omega, P)$ at which $f(\omega, P)$ does not execute all cycles in $\mathcal{C}_\tau(\omega, P)$.[6]

Select a "minimal" problem: Let $t \coloneqq \min_{(\omega,P)} \rho(\omega, P)$. Then $f \neq \varphi$ implies that $t \leq n$. Among all problems in $\{(\omega, P) \in \mathcal{A} \times \mathcal{P}^N \mid \rho(\omega, P) = t\}$, let (ω, P) be one whose *size* is smallest. Hence, for any problem (ω', P'),

$$\text{either (i) } t < \rho(\omega', P') \text{ or (ii) } \rho(\omega', P') = t \text{ and } s(\omega, P) \leq s(\omega', P').$$

Since $\rho(\omega, P) = t \leq n$, $f(\omega, P)$ executes all cycles in $\bigcup_{\tau=1}^{t-1} \mathcal{C}_\tau(\omega, P)$, but it does not execute some cycle in $\mathcal{C}_t(\omega, P)$. Let N^t and O^t be the sets of agents and objects, respectively, that are remaining at step t of $\text{TTC}(\omega, P)$. Let $C \in \mathcal{C}_t(\omega, P)$ be a cycle which is not executed by $f(\omega, P)$. Suppose that

$$C = (i_0, o_1, i_1, o_2, \ldots, o_{k-1}, i_{k-1}, o_k, i_k = i_0).$$

Note that, by the definition of TTC, for each agent $i_\ell \in N(C)$, $o_{\ell+1} = \varphi_{i_\ell}(\omega, P)$ is agent i_ℓ's most-preferred object in O^t at P_{i_ℓ}. Thus,

$$\text{for all } i \in N(C), \quad \varphi_i(\omega, P)\; R_i\; f_i(\omega, P). \tag{1}$$

Because $f(\omega, P)$ does not execute C, there is an agent $i_\ell \in N(C)$ such that $o_{\ell+1} \neq f_{i_\ell}(\omega, P)$. Without loss of generality, let $i_\ell = i_k$ $(= i_0)$. Thus, (1) implies that $o_1\, P_{i_k}\, f_{i_k}(\omega, P)$. If $|N(C)| = k = 1$, then $C = (i_0, o_1, i_1 = i_0)$ and $\omega_{i_1} = o_1\, P_{i_1}\, f_{i_1}(\omega, P)$, which violates *individual rationality* of f. Thus, $|N(C)| \geq 2$.

Claim 1. For each $i_\ell \in N(C)$,

(a) $o_{\ell+1}$ and o_ℓ are "adjacent" in P_{i_ℓ}, i.e., $\{o \in O \setminus \{o_\ell, o_{\ell+1}\} \mid o_{\ell+1}\, P_{i_\ell}\, o\, P_{i_\ell}\, o_\ell\} = \emptyset$; and

[5] We assume that, if $\text{TTC}(\omega, P)$ terminates before step t, then $\mathcal{C}_t(\omega, P) = \emptyset$.

[6] Note that, for each problem (ω, P), (i) $\rho(\omega, P) \leq n + 1$, and (ii) $\rho(\omega, P) = n + 1$ if and only if $f(\omega, P) = \varphi(\omega, P)$.

(b) $\varphi_{i_\ell}(P,\omega) = o_{i_\ell}$.

Proof of Claim 1. First consider agent i_k. Toward contradiction, suppose that (a) fails, i.e., there exists $o \in O \setminus \{o_1, o_k\}$ such that $o_1 \ P_{i_k} \ o \ P_{i_k} \ o_k$. Recall that $\omega_{i_k} = o_k$. Let P'_{i_k} be the truncation of (ω_{i_k}, P_{i_k}) at o_1, i.e., $P'_{i_k} : \ldots, o_1, o_k, \ldots$. Let $P' := (P'_{i_k}, P_{-i_k})$. Then $s(\omega, P') < s(\omega, P)$. Also note that by the definition of TTC, induced cycles remain unchanged, i.e., for each τ, $\mathcal{C}_\tau(\omega, P') = \mathcal{C}_\tau(\omega, P)$. By the choice of (ω, P), $s(\omega, P') < s(\omega, P)$ implies that $\rho(\omega, P') > \rho(\omega, P) = t$. Thus, $f(\omega, P')$ executes all cycles in $\bigcup_{\tau=1}^{t} \mathcal{C}_\tau(\omega, P') = \bigcup_{\tau=1}^{t} \mathcal{C}_\tau(\omega, P)$. Since $C \in \mathcal{C}_t(\omega, P)$, we see that $f(\omega, P')$ executes C. Thus, $f_{i_k}(\omega, P') = o_1$, which contradicts *truncation-proofness* of f. Thus, (a) holds for agent i_k. By (1) and *individual rationality* of f, we must have $f_{i_k}(\omega, P) = o_k$. Thus, (b) also holds for agent i_k.

Now consider agent i_{k-1}. Because $f_{i_k}(\omega, P) = o_k$ and o_k is i_{k-1}'s most-preferred object in O^t at $P_{i_{k-1}}$, we must have $o_k \ P_{i_{k-1}} \ f_{i_{k-1}}(\omega, P)$. Therefore, a similar argument shows that $\{o \in O \setminus \{o_{k-1}, o_k\} \mid o_k \ P_{i_{k-1}} \ o \ P_{i_{k-1}} \ o_{k-1}\} = \emptyset$ and $f_{i_{k-1}}(\omega, P) = o_{k-1}$. That is, conditions (a) and (b) also hold for agent i_{k-1}. Proceeding by induction, one can show that conditions (a) and (b) hold for each agent $i_\ell \in N(C)$. ■

Claim 1, which invokes only *individual rationality* and *truncation-proofness*, implies that, when restricted to the agents in $N(C)$, the problem (ω, P) looks as follows (with agents' endowments underlined):

P_{i_1}	P_{i_2}	$\cdots$	$P_{i_{k-1}}$	P_{i_k}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
o_2	o_3	$\cdots$	o_k	o_1
$\underline{o_1}$	$\underline{o_2}$	$\cdots$	$\underline{o_{k-1}}$	$\underline{o_k}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$

Now consider the problem $(\overline{\omega}, P)$, where $\overline{\omega} := \omega^{i_1 i_2}$ is the initial allocation obtained from ω by letting agents i_1 and i_2 swap their endowments. The following claim says that, for each step $\tau \in \{1, \ldots, t-1\}$, every cycle that obtains under TTC(ω, P) also obtains under TTC$(\overline{\omega}, P)$.

Claim 2. For each $\tau \in \{1, \ldots, t-1\}$, $\mathcal{C}_\tau(\omega, P) \subseteq \mathcal{C}_\tau(\overline{\omega}, P)$.

The intuition behind Claim 2 is as follows. Each cycle in $\bigcup_{\tau=1}^{t-1} \mathcal{C}_\tau(\omega, P)$ involves only agents in $N \backslash N^t$, and each agent $i \in N \backslash N^t$ has the same endowment and the same preferences at the two problems (ω, P) and $(\overline{\omega}, P)$. Thus, $\mathcal{C}_1(\omega, P) \subseteq \mathcal{C}_1(\overline{\omega}, P)$. The remaining inclusions then follow from a recursive argument. The formal proof is given at the end of this subsection.

Claim 2 implies that, at $\varphi(\overline{\omega}, P)$, no agent $i_\ell \in N(C)$ is assigned an object that she prefers to $o_{\ell+1}$, as any such object is assigned to someone else via some cycle in $\bigcup_{\tau=1}^{t-1} \mathcal{C}_\tau(\overline{\omega}, P)$. Thus, by the definition of TTC, the cycles $C' := (i_1, o_2, i_1)$ and $C'' := (i_0, o_1, i_2, o_3, \ldots, o_k, i_k = i_0)$ must clear at some steps $\tau' \leq t$ and $\tau'' \leq t$, respectively, of TTC$(\overline{\omega}, P)$. That is, $C', C'' \in \bigcup_{\tau=1}^{t} \mathcal{C}_\tau(\overline{\omega}, P)$.

Additionally, Claim 2 and the fact that $\rho(\overline{\omega}, P) \geq t$ imply that, at $f(\overline{\omega}, P)$, agent i_1 is not assigned an object that she prefers to $\overline{\omega}_{i_1} = o_2$, as any such object is assigned to someone else via some cycle in $\bigcup_{\tau=1}^{t-1} \mathcal{C}_\tau(\overline{\omega}, P)$. Thus, *individual rationality* of f implies that $f_{i_1}(\overline{\omega}, P) = o_2 \, P_{i_1} \, f_{i_1}(\omega, P)$. By *endowments-swapping-proofness* of f, $o_3 \, P_{i_2} \, f_{i_2}(\overline{\omega}, P)$. Furthermore, $f_{i_2}(\overline{\omega}, P) \neq o_2$ implies that $o_2 \; P_{i_2} \; f_{i_2}(\overline{\omega}, P)$.

Let P'_{i_2} be the truncation of $(\overline{\omega}_{i_2}, P_{i_2})$ at o_3, i.e., $P'_{i_2} : \ldots, o_3, o_1, o_2, \ldots$. Let $P' := (P'_{i_2}, P_{-i_2})$. Then, for the agents in $N(C)$, the problem $(\overline{\omega}, P')$ looks as follows (with agents' endowments underlined):

P'_{i_1}	P'_{i_2}	$\cdots$	$P'_{i_{k-1}}$	P'_{i_k}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\underline{o_2}$	o_3	$\cdots$	o_k	o_1
o_1	$\underline{o_1}$	$\cdots$	$\underline{o_{k-1}}$	$\underline{o_k}$
$\vdots$	o_2	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$

Observe that $s(\overline{\omega}, P') < s(\omega, P)$. Therefore, the choice of (ω, P) implies that $\rho(\overline{\omega}, P') > \rho(\omega, P) = t$. Thus, $f(\overline{\omega}, P')$ executes all cycles in $\bigcup_{\tau=1}^{t} \mathcal{C}_\tau(\overline{\omega}, P')$. By the definition of TTC, the algorithms TTC$(\overline{\omega}, P')$ and TTC$(\overline{\omega}, P)$ generate and execute the same cycles, i.e., for each step τ, $\mathcal{C}_\tau(\overline{\omega}, P') = \mathcal{C}_\tau(\overline{\omega}, P)$. In particular, $f(\overline{\omega}, P')$ executes C''. However, this means that $f_{i_2}(\overline{\omega}, P') = o_3$, a violation of *truncation-proofness*. This completes the proof of Theorem 1 under the assumption that Claim 2 holds.

To prove Claim 2, we prove the following stronger claim.[7]

Claim 3. For each $\tau \in \{1, \ldots, t-1\}$, the following statements hold:

$S_1(\tau)$: $\mathcal{C}_\tau(\omega, P) \subseteq \mathcal{C}_\tau(\overline{\omega}, P)$; and

$S_2(\tau)$: $\overline{C} \in \mathcal{C}_\tau(\overline{\omega}, P) \setminus \mathcal{C}_\tau(\omega, P)$ implies that $O(\overline{C}) \subseteq O^t$.

[7] To prove Claim 2, some additional care is needed to show that, for any step τ, any additional cycle that clears during TTC$(\overline{\omega}, P)$ but not TTC(ω, P) does not "interfere" with the execution of the remaining cycles in $\bigcup_{\tau+1}^{t-1} \mathcal{C}_\tau(\omega, P)$. This is the content of the second part of Claim 3.

Proof of Claim 3. Suppose otherwise. We start by introducing some notation. Let τ be the earliest step at which $S_1(\tau)$ or $S_2(\tau)$ fails. Let $O^\tau(\omega, P)$ and $O^\tau(\overline{\omega}, P)$ denote the sets of objects remaining at step τ of $\text{TTC}(\omega, P)$ and $\text{TTC}(\overline{\omega}, P)$, respectively. Let $N^\tau(\omega, P)$ and $N^\tau(\overline{\omega}, P)$ denote the corresponding sets of agents. For any nonempty subset $X \subseteq O$, let $\text{top}_{P_i}(X)$ denote the most-preferred object in X at P_i.[8]

The choice of τ implies that, for each $\tau' < \tau$, $S_1(\tau')$ and $S_1(\tau')$ are both true. Therefore,

$$O^\tau(\overline{\omega}, P) \subseteq O^\tau(\omega, P) \text{ and } O^\tau(\overline{\omega}, P)\backslash O^t = O^\tau(\omega, P)\backslash O^t.$$

Let $i \in N^\tau(\omega, P)\backslash N^t$ $(= N^\tau(\overline{\omega}, P)\backslash N^t)$. Because $\tau < t$, the definition of TTC implies that agent i prefers $\varphi_i(\omega, P) \in O^\tau(\omega, P)\backslash O^t$ to any object in O^t. Thus, $\text{top}_{P_i}(O^\tau(\omega, P)) \in O^\tau(\omega, P)\backslash O^t$. It follows that, for each $i \in N^\tau(\omega, P)\backslash N^t$,

$$\text{top}_{P_i}(O^\tau(\omega, P)) = \text{top}_{P_i}(O^\tau(\omega, P)\backslash O^t) = \text{top}_{P_i}(O^\tau(\overline{\omega}, P)\backslash O^t) = \text{top}_{P_i}(O^\tau(\overline{\omega}, P)). \tag{2}$$

In other words, at step τ, each agent $i \in N^t(\omega, P)\backslash N^t$ points to the same object in $\text{TTC}(\omega, P)$ and in $\text{TTC}(\overline{\omega}, P)$. We now show that $S_1(\tau)$ holds. Let $\tilde{C} \in \mathcal{C}_\tau(\omega, P)$ and $i \in N(\tilde{C})$. Then agent i points to $\varphi_i(\omega, P)$ on $\tilde{C}$. Because $\tau < t$, we have that $i \in N^\tau(\omega, P)\backslash N^t$. Thus, by (2), (i) agent i also points to $\varphi_i(\omega, P)$ at step τ of $\text{TTC}(\overline{\omega}, P)$. Furthermore, $i \notin N^t$ implies that (ii) $\omega_i = \overline{\omega}_i$. Since (i) and (ii) hold for each agent $i \in N(\tilde{C})$, we have that $\tilde{C} \in \mathcal{C}_\tau(\overline{\omega}, P)$. Thus, $S_1(\tau)$ holds, which means that $S_2(\tau)$ fails.

Because $S_2(\tau)$ fails, there is a cycle $\overline{C} \in \mathcal{C}_\tau(\overline{\omega}, P)\backslash\mathcal{C}_\tau(\omega, P)$ such that $O(\overline{C}) \nsubseteq O^t$ and, hence, $N(\overline{C}) \nsubseteq N^t$. Let $j_0 \in N(\overline{C})\backslash N^t$. Then $N(\overline{C}) \subseteq N^\tau(\overline{\omega}, P) \subseteq N^\tau(\omega, P)$, which means that $j_0 \in N^\tau(\omega, P)\backslash N^t$. Let agent j_0 point to object x_1 on $\overline{C}$. By (2), $x_1 \in O(\overline{C})\backslash O^t$, which means that the owner of x_1 (at ω and $\overline{\omega}$) is an agent $j_1 \in N(\overline{C})\backslash N^t$. Repeating the above argument, we show that, on $\overline{C}$, agent j_1 points to an object $x_2 \in O(\overline{C})\backslash O^t$ which is owned (at ω and $\overline{\omega}$) by an agent $j_2 \in N(\overline{C})\backslash N^t$. A recursive argument shows that all agents on $\overline{C}$ must belong to $N\backslash N^t$. Hence, $N(\overline{C}) \subseteq N^\tau(\omega, P)\backslash N^t$. By (2), (i) every agent on $N(\overline{C})$ points to the same object at step τ during $\text{TTC}(\omega, P)$ and $\text{TTC}(\overline{\omega}, P)$. Moreover, (ii) every agent in $N(\overline{C})$ is endowed with the same object at ω and $\overline{\omega}$. Thus, (i) and (ii) imply that $\overline{C} \in \mathcal{C}_\tau(\omega, P)$, a contradiction. ■

4 Discussion

Recently, Chen et al. (2024) established that the uniqueness results of Fujinaka and Wakayama (2018) and Ekici (2024) both remain true if *strategy-proofness* is weakened to *truncation-*

[8] Formally, $\text{top}_{P_i}(X) \in X$ and, for all $o \in X$, $\text{top}_{P_i}(X)\ R_i\ o$.

invariance.[9] That is, they show that TTC is characterized by the following sets of properties:

1. *individual rationality*, *truncation-invariance*, and *endowments-swapping-proofness*; and
2. *individual rationality*, *truncation-invariance*, and *pair-efficiency*.

While Theorem 1 shows that the uniqueness result of Fujinaka and Wakayama (2018) can be refined by relaxing *strategy-proofness* to *truncation-proofness*, the uniqueness result of Ekici (2024) does not permit a similar refinement. The following example gives a rule, different from TTC, that still satisfies *individual rationality*, *truncation-proofness*, and *pair-efficiency*.[10]

Example 1 (*Individual rationality*, *truncation-proofness*, and *pair-efficiency* $\not\Rightarrow$ TTC)**.** Let $N = \{1, 2, 3\}$. Let (ω^*, P^*) be a problem with $\omega^* = (o_1, o_2, o_3)$ and

$$P_1^* : o_2, o_1, o_3; \quad P_2^* : o_3, o_2, o_1; \quad P_3^* : o_1, o_3, o_2.$$

Let f be the rule defined as follows:

$$f(\omega, P) = \begin{cases} \omega^*, & \text{if } (\omega, P) = (\omega^*, P^*) \\ \varphi(\omega, P), & \text{otherwise.} \end{cases}$$

Clearly, φ is *pair-efficient* and *individually rational*. It is straightforward to show that f is *truncation-proof*. However, f is not *truncation-invariant*: If $P_1' : o_2, o_3, o_1$, then

$$f_1(\omega^*, (P_1', P_{-1}^*)) = o_2 \; P_1^* \; o_1 = f_1(\omega^*, P^*),$$

even though P_1' agrees with P_1^* on $\{o \in O \mid o \; P_1' \; f_1(\omega^*, P^*)\}$. Similarly, f is not *endowments-swapping-proof* because agents 1 and 2 prefer to swap their endowments at (ω^*, P^*). $\diamond$

Example 1 demonstrates that, in the presence of *individual rationality* and *pair-efficiency*, *truncation-proofness* is strictly weaker than *truncation-invariance*.[11] It also sheds some light on the importance of our proof technique, whereby we select a problem that is "minimal" according to *both* similarity and size. Chen et al. (2024) showed that, under *truncation-invariance*, the original approach of Sethuraman (2016) (see also Ekici and Sethuraman, 2024)—which exploits only the size of a problem—is sufficient to pin down TTC. Example 1 highlights the difficulty

[9]A rule f is **truncation-invariant** if, for each problem (ω, P), each $i \in N$, and each $P_i' \in \mathcal{P}$, $f_i(\omega, (P_i', P_{-i})) = f_i(\omega, P)$ whenever P_i' agrees with P_i on $\{o \in O \mid o \; P_i' \; f_i(\omega, P)\}$.

[10]This example first appeared in an early draft of Coreno and Feng (2024).

[11]In contrast, *truncation-proofness* and *truncation-invariance* are equivalent in the presence of *individual rationality* and either of *endowments-swapping-proofness* (Theorem 1) or *Pareto efficiency* (Coreno and Feng, 2024).

in adapting this argument under *truncation-proofness*. The difficulty arises because *truncation-proofness* precludes agents from manipulating in only one direction: it defends against manipulations from a preference relation P_i to a truncation P'_i of (ω_i, P_i), but it does not prevent manipulations from P'_i back to P_i.[12]

Our analysis suggests a promising direction for future research. Given the wide variety of rules satisfying *individual rationality*, *truncation-proofness*, and *pair-efficiency*, a complete characterization of this entire class would be a significant contribution. Clearly, the rule f of Example 1 is unsatisfactory, as it is Pareto-dominated by φ. It would be interesting to know whether this class admits other appealing rules.

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[12] The difference between these two types of manipulations is significant. For instance, the *efficiency-adjusted deferred acceptance* rule is *truncation-proof* but it does not prevent manipulations from a truncation P'_i back to P_i. See Shirakawa (2024) for details.

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