Axiomatic Characterizations of Draft Rules

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SLMath
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Ex-Spouses Go to Court to Split Beanie Babies

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LAS VEGAS — A divorced couple who couldn't agree on how to split up their Beanie Baby collection were ordered by a judge Friday to divide up the babies one by one in a courtroom.

Maple the Bear was the first to go.

[…]

Frances and Harold Mountain divorced four months ago. According to the divorce decree, the parties were supposed to divide their Beanie Baby collection, estimated to be worth between $2,500 and $5,000.

But they failed to split up the toys by themselves. After Harold Mountain filed a motion to get his share of the toys, the judge said he had had enough.

"So I told them to bring the Beanie Babies in, spread them out on the floor, and I'll have them pick one each until they're all gone."

Drafts

- A simple and widely-used *round-robin* allocation procedure:
  - *agents* take turns to choose items from a set of heterogeneous and indivisible *objects*.
  - within each *round*, each agent selects a single object in some fixed priority order.
- It sees applications in divorce settlements (Brams et al., 2015), course allocation (Budish and Cantillon, 2012), estate division (Heath, 2018), the assignment of tasks to workers, etc.
- Its most prominent and economically important application is in the allocation of recruits to teams in professional sports leagues.
- There it is universally known as the *draft*.

Drafts in sports

- The draft was first proposed in 1935 by Bert Bell, an owner of the National Football League (NFL)'s Philadelphia Eagles, a perennial underperformer at that time.
- The proposal stipulated that underperforming teams would get higher *priority*.
- Choosing a player granted a team the exclusive right to negotiate with them.
- The main rationale was to give weaker teams the chance to sign talented players and build more competitive rosters.

Drafts in sports

- Most other (closed) sports leagues have now adopted a draft.
- Universally, the draft's main stated goal is to maintain *competitive balance* among the league's members.
- To that end, the priority ordering in the draft is determined by final league standings in the preceding season with worse performing teams choosing earlier.

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- To that end, the priority ordering in the draft is determined by final league standings in the preceding season with worse performing teams choosing earlier.
- Drafts are economically important:
  - A league's competitive balance is an important determinant of profitability through ticket and merchandise sales, TV rights, sponsorships, etc.
  - Each of the major North American sports leagues boasts multi-billion dollar revenue, massive TV deals, and rapidly rising franchise values.
  - Cal Golden Bears have produced two #1 draft picks, including Jared Goff in 2016, who signed a four-year deal with the LA Rams worth $27.9 million.

Plan and main questions

- We consider the draft as a (centralized) allocation rule, and we analyze it using the axiomatic approach.
- What desirable properties does the draft satisfy? And which of them help to promote competitive balance?
- Could there be better mechanisms that help redress competitive imbalances?

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  - here we obtain RP as a consequence of the other properties.

- Although drafts are not *strategy-proof (SP)*...
  - ... no allocation rule satisfies SP and the competitive-balance properties, RP and EF1.
  - ... they satisfy a weaker incentive property that we call *maxmin strategy-proofness.*

Related literature

Theoretical studies of the draft:

- Rottenberg (1956), Kohler and Chandrasekaran (1971), Brams and Straffin (1979), Brams and King (2005), Budish and Cantillon (2012), Caragiannis et al. (2019).

Multiple-object allocation problems:

- Pápai (2000; 2001), Ehlers and Klaus (2003), Hatfield (2009), Budish (2011), Biró et al. (2022a; 2022b).

Model: Allocations

- $N = \{1, \ldots, n\}$ is a set of *agents*.
- $\mathbb{O}$ is a set of *(potential) objects.*
- $2^{\mathbb{O}}$ is the family of sets of *available objects.*
- Given $X \subseteq \mathbb{O}$, an *X-allocation* is a profile $A = (A_i)_{i \in N}$ of disjoint subsets of X.

Model: Preferences

- Each agent i reports *strict preferences* $\succeq_i$ over $\mathbb{O}$.
  - $x \succeq_i y$ means ($x \succ_i y$ or $x = y$).
  - useful to write, e.g., $\succeq_i = a, b, c, \ldots$ to specify agent i's preferences.
  - $\succeq = (\succeq_i)_{i \in N}$ denotes a *preference profile*.
- The *pairwise dominance extension* $\succeq_i^{PD}$ of $\succeq_i$ is the partial order on $2^{\mathbb{O}}$ defined as follows: for all $S, T \subseteq \mathbb{O}$, $S \succeq_i^{PD} T$ iff there is an injection $\mu : T \to S$ such that $\mu(x) \succeq_i x$ for all $x \in T$.

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Example

If $a \succ_i b \succ_i c$, then

$$\{a, b, c\} \succ_i^{PD} \{a, b\} \succ_i^{PD} \{a, c\} \succ_i^{PD} \{a\}, \{b, c\} \succ_i^{PD} \{b\} \succ_i^{PD} \{c\} \succ_i^{PD} \emptyset,$$

but $\{a\}$ and $\{b, c\}$ are not comparable.

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Remark

The pairwise dominance extension $\succeq_i^{PD}$ is equivalent to both the *responsive set extension* and the *additive utility extension*. That is,

$$\succeq_i^{PD} = \bigcap_{R_i \in \mathcal{R}(\succeq_i)} R_i = \bigcap_{R_i \in \mathcal{A}(\succeq_i)} R_i,$$

where $\mathcal{R}(\succeq_i)$ (resp. $\mathcal{A}(\succeq_i)$) is the set of responsive (resp. additive) preference relations on $2^{\mathbb{O}}$ consistent with the relation $\succeq_i$ on $\mathbb{O}$.

Model: Allocation rules

- A *problem* $(\succeq, X)$ comprises a preference profile $\succeq$ and a set $X \subseteq \mathbb{O}$.
- An *allocation rule* φ maps each problem $(\succeq, X)$ to an X-allocation $\varphi(\succeq, X)$.
- A *priority* π is a linear order on N.
  - $i \pi j$ means *agent i has higher priority than j*.
- The *draft rule associated with π*, φ^π, assigns each agent her best remaining object, one at a time, in the order prescribed by π; the process repeats once all agents have received an object.[a]

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[a] i.e., φ^π maps each problem $(\succeq, X)$ to the allocation $\varphi^\pi(\succeq, X)$ defined as follows:

- Let $f^\pi : \mathbb{N} \to N$ denote the *picking sequence associated with π*: i.e., if $i_1 \pi \cdots \pi i_n$, then $(f^\pi(t))_{t \in \mathbb{N}} = (i_1, \ldots, i_n, i_1, \ldots, i_n, \ldots)$.
- Recursively define a sequence $(s_t)_{t=1}^{|X|}$ of selections by $s_1 = \text{top}_{f^\pi(1)}(X)$ and, for each $t = 2, \ldots, |X|$, $s_t = \text{top}_{f^\pi(t)}(X \setminus \{s_1, \ldots, s_{t-1}\})$.
- For each $i \in N$, set $\varphi_i^\pi(\succeq, X) = \{s_t \mid f^\pi(t) = i\}$.

Properties: Fairness

An allocation rule φ is

(1) *respectful of a priority (RP)* if there exists a priority π such that for each problem $(\succeq, X)$ and each agent i,

$$\varphi_i(\succeq, X) \succeq_i^{PD} \varphi_j(\succeq, X) \text{ whenever } i\pi j.$$

(2) *envy-free up to one object (EF1)* if
for any problem $(\succeq, X)$ and any agents $i, j \in N$, there exists $S \subseteq \varphi_j(\succeq, X)$ such that $|S| \leq 1$ and

$$\varphi_i(\succeq, X) \succeq_i^{PD} \varphi_j(\succeq, X) \setminus S.$$

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- RP and EF1 are relaxations of *envy-freeness (EF)*.
- RP is a form of *no justified envy*: if i (possibly) envies j (i.e., $\varphi_i(\succeq, X) \not\succeq_i^{PD} \varphi_j(\succeq, X)$), then $j\pi i$.
- both properties are closely related to competitive balance.

Properties: Efficiency and solidarity

An allocation rule φ is

(3) *efficient (EFF)* if
for each problem $(\succeq, X)$, $\varphi(\succeq, X)$ is not Pareto dominated by any X-allocation wrt $\succeq^{PD}$.

(4) *non-wasteful (NW)* if it always assigns all available objects:
for each problem $(\succeq, X)$, $\bigcup_{i \in N} \varphi_i(\succeq, X) = X$.

(5) *resource monotonic (RM)* if
for any preference profile $\succeq$ and $X, X' \subseteq \mathbb{O}$,

$$X \supseteq X' \implies \varphi_i(\succeq, X) \succeq_i^{PD} \varphi_i(\succeq, X') \text{ for all } i \in N.$$

Properties of draft rules

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Characterization 1

An allocation rule φ satisfies RP, EF1, NW, and RM iff
φ is a draft rule, i.e., there exists a priority π such that $\varphi = \varphi^\pi$.

Proof sketch: Step 1

Lemma 1

If φ satisfies RP-π and EF1, then there is an agent $i \in N$ such that

$$\begin{aligned} & |\varphi_j(\succeq, X)| = |\varphi_i(\succeq, X)| \text{ whenever } j\pi i \\ \text{and} \quad & |\varphi_j(\succeq, X)| = |\varphi_i(\succeq, X)| - 1 \text{ whenever } i\pi j \text{ and } i \neq j. \end{aligned}$$

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- If $i_1 \pi \cdots \pi i_n$, then RP-π implies

$$|\varphi_{i_1}(\succeq, X)| \geq |\varphi_{i_2}(\succeq, X)| \geq \cdots \geq |\varphi_{i_n}(\succeq, X)|.$$

- By EF1, for all $i, j \in N$ it holds that

$$|\varphi_i(\succeq, X)| - |\varphi_j(\succeq, X)| \leq 1.$$

Proof sketch: Step 2

Lemma 2

Suppose φ satisfies RM and that $\varphi(\succeq, X) = \varphi^{\pi}(\succeq, X)$. If $x \in \mathbb{O} \backslash X$ is such that, for all $i \in N$,

$$y \succ_i x \text{ for each } y \in \varphi_i(\succeq, X),$$

then

$$\varphi_i(\succeq, X) \subseteq \varphi_i(\succeq, X \cup \{x\}) \text{ for each } i \in N.$$

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then

$$\varphi_i(\succeq, X) \subseteq \varphi_i(\succeq, X \cup \{x\}) \text{ for each } i \in N.$$

- i.e., each agent's assigned bundle in the smaller problem is included in her bundle in the larger problem.
- i_1 must retain her favorite object s_1;
  otherwise, $\varphi_{i_1}(\succeq, X \cup \{x\}) \not\succeq_{i_1}^{PD} \varphi_{i_1}(\succeq, X)$, violating RM.
- Similarly, i_2 must retain her favorite object, etc.

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- Consider $S_1 = \{s_1\}$:
  - Step 1 and NW imply $\varphi_{i_1}(\succeq, S_1) = \{s_1\} = \varphi^\pi_{i_1}(\succeq, S_1)$.
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- Consider $S_2 = \{s_1, s_2\}$:
  - Step 2 implies $\varphi_{i_1}(\succeq, S_1) = \{s_1\} \subseteq \varphi_{i_1}(\succeq, S_2)$.
  - By Step 1 and NW, $\varphi_{i_2}(\succeq, S_2) = \{s_2\}$.
  - Hence, $\varphi(\succeq, S_2) = \varphi^\pi(\succeq, S_2)$.

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  - By Step 1 and NW, $\varphi_{i_2}(\succeq, S_2) = \{s_2\}$.
  - Hence, $\varphi(\succeq, S_2) = \varphi^\pi(\succeq, S_2)$.
- ... and so on.

RP and EF1 promote competitive balance

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- RP guarantees that no agent envies any agent with lower priority.
  - Allows leagues to support weaker teams.
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  - *Serial dictatorships* also satisfy RP (as well as efficiency and *strategy-proofness*).
  - But low-priority agents may envy high-priority ones *severely*.
- EF1 limits the extent to which low-priority agents can envy high-priority agents.
  - Ensures weaker teams not favored too heavily.
  - Prevents "over-correction" of the competitive balance and large swings in team rankings.
  - Limits incentives to *tank*.

Properties: Incentives

An allocation rule φ is

(6) *strategy-proof (SP)* if
for each problem $(\succeq, X)$, each agent i, and each report $\succeq_i'$,

$$\varphi_i(\succeq, X) \succeq_i^{PD} \varphi_i\left((\succeq_i', \succeq_{-i}), X\right).$$

(7) *weakly strategy-proof (WSP)* if
for each problem $(\succeq, X)$ and each agent i, there is no report $\succeq_i'$ such that

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Unfortunately, draft rules are not even weakly strategy-proof:

- an agent can benefit by ranking popular objects above unpopular ones she likes more.

Impossibility results

No allocation rule can meaningfully improve upon the draft's properties.

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If $n = 2$, then no allocation rule satisfies EF1, NW, and SP.

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Impossibility 3

If $n = 2$, then no allocation rule satisfies EF1, NW, and SP.

- In Impossibility 1, EF1, NW, and WSP, are indispensable. Does there exist an allocation rule satisfying EF1, NW, and WSP?
- Does Impossibility 3 extend to $n \geq 2$? We think so, but case-checking becomes unwieldy.

Maxmin strategy-proofness

- Although draft rules are not WSP, they satisfy *maxmin strategy-proofness (MSP).*

- i.e., if an agent evaluates choices based on their worst-possible outcome (i.e., the outcome that would arise if playing against adversarial opponents), then truth-telling is optimal.

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Theorem

Every draft rule φ^π is MSP: for each $X \subseteq \mathbb{O}$, each $i \in N$, each true preference relation $\succeq_i$, and each additive u_i consistent with $\succeq_i$,

$$\succeq_i \in \arg\max_{\succeq'_i} \left[\min_{\succeq'_{-i}} u_i \left(\varphi^\pi_i \left(\left(\succeq'_i, \succeq'_{-i} \right), X \right) \right) \right].$$

Extension: Variable Populations

- $\mathbb{N} = \{1, 2, \dots\}$ is a set of *potential agents*.
- $\mathcal{N} = \{N \subseteq \mathbb{N} \mid 0 < |N| < \infty\}$ denotes all possible sets of agents.
- A problem is a triple $(N, X, \succeq)$, where $N \in \mathcal{N}$, $X \subseteq \mathbb{O}$, and $\succeq$ is a preference profile on X.

Properties: Consistency

An allocation rule φ is

(8) *(population) consistent (CON)* if, for any problem $(N, X, \succeq)$ and any nonempty set $N' \subsetneq N$, and any $i \in N \backslash N'$,

$$\varphi_i \left(N \backslash N', X \backslash X', \succeq|_{X \backslash X'} \right) = \varphi_i \left(N, X, \succeq \right),$$

where $X' = \bigcup_{i \in N'} \varphi_i \left(N, X, \succeq \right)$.

(9) *top-object consistent (T-CON)* if, for any problem $(N, X, \succeq)$ and any agent $i \in N$,

$$\varphi_i \left(N, X \backslash X', \succeq|_{X \backslash X'} \right) = \varphi_i \left(N, X, \succeq \right) \backslash X',$$

where $X' = \bigcup_{i \in N : \varphi_i(\succeq, X) \neq \emptyset} \left\{ \text{top}_{\succeq_i} \left(\varphi_i \left(N, X, \succeq \right) \right) \right\}$.

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- CON is a well-established property (e.g., Ergin, 2000; Thomson, 2011): it guarantees robustness to nonsimultaneous processing of the *agents*.
- T-CON gives a similar guarantee: it ensures a form of robustness to nonsimultaneous processing of the *objects*.

Properties: Neutrality

An allocation rule is

(10) *neutral (NEU)* if, for any problem $(N, X, \succeq)$, any set $X' \subseteq \mathbb{O}$, and any bijection $\sigma : X \to X'$,

$$\sigma\left(\varphi\left(N, X, \succeq\right)\right) = \varphi\left(N, X', \succeq^{\sigma}\right),$$

where $\sigma\left(\varphi\left(N, X, \succeq\right)\right) = \left(\sigma\left(\varphi_i\left(N, X, \succeq\right)\right)\right)_{i \in N}$ and $\succeq^{\sigma}$ is the profile obtained from $\succeq$ by relabelling the objects according to σ.[1]

[1] i.e., $\succeq^{\sigma}$ is the profile on X' such that, for all $i \in N$,

$$\text{for all } x, y \in X,\ x \succeq_i y \iff \sigma(x) \succeq_i \sigma(y).$$

Properties: Neutrality

An allocation rule is

(10) *neutral (NEU)* if, for any problem $(N, X, \succeq)$, any set $X' \subseteq \mathbb{O}$, and any bijection $\sigma : X \to X'$,

$$\sigma\left(\varphi\left(N, X, \succeq\right)\right) = \varphi\left(N, X', \succeq^{\sigma}\right),$$

where $\sigma\left(\varphi\left(N, X, \succeq\right)\right) = \left(\sigma\left(\varphi_i\left(N, X, \succeq\right)\right)\right)_{i \in N}$ and $\succeq^{\sigma}$ is the profile obtained from $\succeq$ by relabelling the objects according to σ.[1]

- NEU ensures that the outcome of the allocation rule is independent of the "identity" of the objects
(e.g., it rules out the *father-son rule* in the AFL)

- it plays a mostly technical role here, however.

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Characterization 2

An allocation rule φ satisfies EF1, EFF, RM, NEU, CON, and T-CON iff φ is a draft rule, i.e., there exists a priority π such that $\varphi = \varphi^{\pi}$.

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Another Characterization.

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An allocation rule φ satisfies EF1, EFF, RM, NEU, CON, and T-CON iff φ is a draft rule, i.e., there exists a priority π such that $\varphi = \varphi^{\pi}$.

- here a priority is derived even *without* assuming RP.
- the proof consists of two lemmas:

(1) If φ is an allocation rule satisfying EF1, EFF, RM, NEU, and CON, then φ agrees with a serial dictatorship on single-unit problems: i.e., there is a priority π such that $\varphi(N, X, \succeq) = \varphi^{\pi}(N, X, \succeq)$ whenever $|X| \leq |N|$.

(2) Suppose φ and π are such that $\varphi(N, X, \succeq) = \varphi^{\pi}(N, X, \succeq)$ whenever $|X| \leq |N|$. If φ satisfies RM and T-CON, then $\varphi(N, X, \succeq) = \varphi^{\pi}(N, X, \succeq)$ *for all problems*.

Extension: Unacceptable Objects

Setup is the same as the fixed population setup, except:

- each preference relation $\succeq_i$ is defined on $\mathbb{O} \cup \{\omega\}$, where ω is the *null object.*
- the set of *acceptable* objects at $\succeq_i$ is $U(\succeq_i) = \{x \in \mathbb{O} \mid x \succ_i \omega\}$.
- the *draft rule associated with* π is the allocation rule φ^π which assigns agents their top-ranked remaining (possibly null) object, one at a time, in the order prescribed by π.

Properties of Allocation Rules

An allocation rule φ is

(1) *non-wasteful (NW)* if
for any problem $(\succeq, X)$, all *acceptable* objects are allocated.

(2) *individually rational (IR)* if
for any problem $(\succeq, X)$, no agent is assigned an unacceptable object.

(3) *truncation invariant (TI)* if[2]
for any problem $(\succeq, X)$ and each agent $i \in N$,

$$\varphi_i(\succeq, X) = \varphi_i\left((\succeq_i', \succeq_{-i}), X\right)$$

whenever $\succeq_i'$ is a truncation of $\succeq_i$ such that $\varphi_i(\succeq, X) \subseteq U(\succeq_i')$.

[2] TI is implied by IR together with *truncation-proofness (TP)* and *extension-proofness (EP)*.

Characterization

Characterization 3

An allocation rule φ is

- non-wasteful (NW),
- resource monotonic (RM),
- respectful of a priority (RP),
- envy-free up to one object (EF1),
- individually rational (IR), and
- truncation invariant (TI)

if and only if

- φ is a draft rule.

Summary

- Our axiomatic characterizations of the draft suggest that its properties are suitable for redressing competitive imbalances in sports leagues.
- The draft is not strategy-proof, but truth-telling is optimal if agents are maxmin utility maximizers.
- It is impossible to meaningfully improve on the draft's properties.

Thank you!

☺

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